Chirality, Hawking Effect and Black Hole Spectroscopy

Bibhas Ranjan Majhi

S.N.Bose National Centre for Basic Sciences,India

Indian Statistical Institute
Kolkata

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(i) R.Banerjee and B.R.Majhi, *Phys.Rev.D79:064024,2009*;

(ii) R.Banerjee and B.R.Majhi, *Phys.Lett.B675:243,2009*;

(iii) R.Banerjee, B.R.Majhi and E.C.Vagenas, *arXiv:0907.4271*

PLAN OF THE TALK

(i) Brief introduction: 1. Hawking effect; 2. Metric and null coordinates; 3. Chirality conditions.

(ii) Anomaly and Hawking radiation.

(iii) Quantum Tunneling and Hawking radiation.

(iv) Comparison between two methods.

(v) Black hole spectroscopy in tunneling mechanism.

(vi) Final remarks.

Brief introduction

Hawking effect

Classical GTR:

- Black Holes: Objects from which nothing can escape.
- No Hair theorem:Collapse leads to a black hole endowed with small number of macroscopic parameters (mass, charge, angular momentum) with no other free parameters.

Quantum effect:

- Black holes can radiate (Hawking effect, 1975).
- Analogous to thermal black body radiation. (neglecting back scattering)

$\Rightarrow$ Black holes may play a major role in the attempts to shed some light on the nature of QTG (as the role played by atoms in early development of QM).

$\Rightarrow$ Hawking effect draws the main attention .

Several derivations of this effect lead to same answer.

Common feature:

(i)Hawking radiation is determined universally by the horizon properties of a black hole.

(ii)Temp. $T_H = \frac{\hbar\kappa}{2\pi}$; Entropy $S_{BH} = \frac{A}{4\hbar}$,

where $\kappa =$ surface gravity (acceleration near the event horizon as measured at infinity)

$A =$ horizon area.

- Recent widely used approaches are: (a) Chiral anomaly method and (b) tunneling method.

Metric and null coordinates

- BH space-time (static, spherically symmetric and asymptotically flat):
  $ds^2 = F(r)dt^2 - \frac{dr^2}{F(r)} - r^2 d\Omega^{(2)}$.

  Event horizon: $\Rightarrow F(r = r_H) = 0$.

- Well known result: near the event horizon the effective theory reduces to 2-dim. conformal theory whose metric is given by the $(r - t)$-sector of the full metric.

- Introduce null tortoise coordinates:
  $u = t - r^*, v = t + r^*$, where $dr^* = \frac{dr}{F(r)}$.

  $(r - t)$ sector of the full metric: $ds^2 = \frac{F(r)}{2}(dudv + dvdu)$.

Chirality conditions

- Massless Klein-Gordon equation: $g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = \frac{4}{F}\partial_u\partial_v\phi = 0$.

  General solution: $\phi(u,v) = \phi^{(R)}(u) + \phi^{(L)}(v)$;

  $\phi^{(R)}(u)$: right (outgoing) mode, $\phi^{(L)}(v)$: left (ingoing) mode.

  Satisfying chirality condition:
  $\nabla_\mu\phi = \pm\bar{\epsilon}_{\mu\nu}\nabla^\nu\phi = \pm\sqrt{-g}\epsilon_{\mu\nu}\nabla^\nu\phi$;

  $+(-)$: $L(R)$ mode; $\epsilon_{uv} = \epsilon_{tr} = -1$.

- Chirality condition for energy-momentum tensor:
  $T_{\mu\nu} = \pm\frac{1}{2}(\bar{\epsilon}_{\mu\sigma}T^\sigma_\nu + \bar{\epsilon}_{\nu\sigma}T^\sigma_\mu) + \frac{1}{2}g_{\mu\nu}T^\alpha_\alpha$
  $\Rightarrow$ For R mode: $T^{(R)}_{vv} = 0, \quad T^{(R)}_{uu} \neq 0$ and for L mode:
  $u \leftrightarrow v$, $R \rightarrow L$.

Anomaly and Hawking radiation

- Horizon: One directional membrane through which nothing can leak (classically)
- Near the event horizon theory becomes two dimensional with its metric identified with the (r-t) section of the original metric.
- Ingoing (say left moving) modes lost inside the horizon; They cannot contribute to the near horizon theory, hence chiral.
- Hence the L and R modes satisfy the chirality conditions.

- Use of these chirality conditions leads to nonconservation of energy-momentum tensor (at the quantum level)

  $\rightarrow$ the chiral (gravitational) anomaly:

  $\nabla^{\mu} T^{(R)}_{\mu\nu} = \frac{1}{96\pi} \bar{\epsilon}_{\nu\lambda} \nabla^{\lambda} R.$

  Similar expression for L mode can be obtained from the chirality conditions. In this case there is a $(-)$ sign in the RHS.

  However, since L mode is trapped inside the horizon, only the anomaly expression for R mode is sufficient for our analysis.

  This expressions is in covariant form and also only the 2-dim. gravitational quantities of the metric are involved.

  $\rightarrow$ Covariant gravitational anomaly.

- Introduction of chiral (gravitational) anomaly in studing Hawking effect was first discussed by Wilczek and Robinson (**PRL (2005)**).

  They used consistent expression for anomaly (satisfying Wess-Zumino consistency condition and not in covariant form) with covariant boundary condition.

- Later a technically simple (only one Ward identity) and more conceptually correct (covariant expression for anomaly with covariant BC) derivation of Hawking flux was introduced by Banerjee and Kulkarni. (**PRD (2008)**)

  I shall follow Banerjee and Kulkarni approach.

- For $\nu = u$:
  $\partial_r T^{(R)}_{uu} = \frac{F}{96\pi}\partial_r R = \frac{F}{96\pi}\partial_r(F'') = \frac{1}{96\pi}\partial_r(FF'' - \frac{F'^2}{2})$

  Solution: $T^{(R)}_{uu} = \frac{1}{96\pi}\left(FF'' - \frac{F'^2}{2}\right) + C_{uu}$;

  C_{uu} is the integration const. which will be fixed by using the covariant boundary condition (CBC): $T^{(R)}_{uu}(r \to r_H) = 0$.

  $\Rightarrow C_{uu} = \frac{F'^2(r_H)}{192\pi}$.
- Significance of the CBC: A freely falling observer sees a finite amount of flux at the outer horizon; corresponds to the Unruh vacuum (*R.Banerjee and S.Kulkarni*, **PRD (2009)**).

  The condition on the ingoing mode for Unruh vacuum ($T^{(R)}_{vv}(r \to \infty) = 0$) is satisfied by default since due to chirality $T^{(R)}_{vv} = 0$.

- For Hawking effect the observer is at infinity. Taking $r \to \infty$ of $T^{(R)}_{uu}$ we obtain the flux:

$$T^{(R)}_{uu}(r \to \infty) = C_{uu} = \frac{F'^2(r_H)}{192\pi} = \frac{\kappa^2}{48\pi}.$$

$\kappa = \frac{F'(r_H)}{2}$ is the surface gravity of the black hole.

This corresponds to the flux from a black body with the temperature $T_H = \frac{\kappa}{2\pi}$ (in units of $\hbar = 1$).

This is the standard result in Hawking effect.

Quantum tunneling and Hawking radiation

- Inside a black hole virtual pair production occurs.
  Similar to an electron-positron pair creation in a constant electric field.

- It may happen that one of them quantum mechanically tunnels radially just outside the horizon and may be detected at infinity while the other falls to the center of the black hole.

$\Rightarrow$ The existence of horizon is crucial here:completely quantum mechanical effect.

Also, $(r - t)$ sector of orginal metric is important.

- Two methods to discuss tunneling mechanism:(only semiclassical approximation; temp. but no spectrum)

  (i) Hamilton-Jacobi method (*Srinivasan and Padmanabhan*, **PRD (1999)**)

  (ii) Null geodesic method (*Parikh and Wilczek*, **PRL (2000)**)

  Our new results:
  (i) Derivation HJ equation and the modes from the chirality condition.

  (ii) Reformulation of tunneling approach based on these modes.

  (iii) Obtention of the emission spectrum $\rightarrow$ perfect black body spectrum in the semiclassical limit. Temp. is identified by comparing the emission spectrum with that from a perfect black body.

HJ equation and particle modes

- Under the $(r-t)$ sector of metric, the chirality condition:

  $\nabla_\mu \phi = \pm\sqrt{-g}\epsilon_{\mu\nu}\nabla^\nu \phi$

  reduces to:

  $\partial_t \phi(r,t) = \pm F(r)\partial_r \phi(r,t)$.

  $+(-)$: $L(R)$ mode.

- Substitute the WKB ansatz: $\phi(t,r) = e^{-\frac{i}{\hbar}S(t,r)}$;
  $S(t,r)$: Particle action.
  with,

  $$S(t,r) = S_0(t,r) + \sum_{i=1}^{\infty} \hbar^i S_i(t,r).$$

- Equate $\hbar^0$ coefficients on both sides:
  $\partial_t S_0 = \pm F(r)\partial_r S_0$: Semiclassical HJ equation.

- Solution for $S_0(t, r)$:

  Metric is stationary.

  $\rightarrow$ It has time-like Killing vector.

  $\rightarrow$ There will be constant of motion (ω) corresponding to the timelike Killing vector.

  Choose: $S_0(t, r) = \omega t + \tilde{S}_0(r)$; ω is identified as the effective energy of the particle at asymptotic infinity.

  Substitute this in semiclassical HJ equation and then solve:

  $S_0(t, r) = \omega(t \pm r^*)$; $r^* = \int \frac{dr}{F(r)}$. (Tortoise coords.)

  $+(-)$: Ingoing (L mode)/ outgoing (R mode) solution.

- Null tortoise coordinates:
  $u = t - r^*;\ v = t + r^*.$

  and so,

  $S_0^{(R)} = \omega u;\ S_0^{(L)} = \omega v.$

  $\Rightarrow S^{(R)} = \omega u$

  $S^{(L)} = \omega v.$

- Modes as defined outside and inside the horizon:

  $\phi_{out}^{(R)} = e^{-\frac{i}{\hbar}\omega u_{out}},\ \phi_{out}^{(L)} = e^{-\frac{i}{\hbar}\omega v_{out}}$

  $\phi_{in}^{(R)} = e^{-\frac{i}{\hbar}\omega u_{in}},\ \phi_{in}^{(L)} = e^{-\frac{i}{\hbar}\omega v_{in}}.$

- Quantum tunneling occurs across the event horizon.
  → Nature of (r, t) coordinates changes.
  → Proper coordinates system which can describe the whole space-time is Kruskal coordinates.

- The relation between Kruskal time (T), space (X) coordinates and Schwarzschild coordinates (t, r) defined inside and outside the horizon:

  $T_{in} = e^{\kappa r^*_{in}} \cosh(\kappa t_{in})$; $X_{in} = e^{\kappa r^*_{in}} \sinh(\kappa t_{in})$

  $T_{out} = e^{\kappa r^*_{out}} \sinh(\kappa t_{out})$; $X_{out} = e^{\kappa r^*_{out}} \cosh(\kappa t_{out})$.

  $\kappa = \frac{F'(r_H)}{2}$: surface gravity.

  (T, X) are defined in such a way that
  in both sectors their nature remains same.

- Metric is identical in both sectors:

$$T_{in} = T_{out};\ X_{in} = X_{out}.$$

$$\Rightarrow t_{in} = t_{out} - \frac{i\pi}{2\kappa};\ r^*_{in} = r^*_{out} + \frac{i\pi}{2\kappa}.$$

$$\Rightarrow u_{in} = t_{in} - r^*_{in} = t_{out} - r^*_{out} - \frac{i\pi}{\kappa} = u_{out} - \frac{i\pi}{\kappa};$$

and $v_{in} = v_{out}$.

- Therefore the relations between the modes defined in both sectors are:

$$\phi^{(R)}_{in} = e^{-\frac{\pi\omega}{\hbar\kappa}}\phi^{(R)}_{out};$$

$$\phi^{(L)}_{in} = \phi^{(L)}_{out}.$$

Observations:

- Observer at infinity ($r \to \infty$) outside the horizon.

• To observe the pair produced at $r < r_H$, one has to write the "in" modes in terms of "out" coordinates.

• The probability of L-mode to go inside is
$$P^{(L)} = \left|\phi_{in}^{(L)}\right|^2 = \left|\phi_{out}^{(L)}\right|^2 = 1.$$

$\Rightarrow$ L-mode trapped inside the BH. [Consistency check]

• Similarly, the prob. of R-mode to go outside is
$$P^{(R)} = \left|\phi_{in}^{(R)}\right|^2 = \left|e^{-\frac{\pi\omega}{\hbar\kappa}}\phi_{out}^{(R)}\right|^2 = e^{-\frac{2\pi\omega}{\hbar\kappa}}.$$

• In the classical limit ($\hbar \to 0$): $P^{(R)} \to 0$ and $P^{(L)} \to 1$.

$\Rightarrow$ Satisfies classical result: BH cannot radiate classically.

- Painleve coordinates also remove the apparent singularity at $r = r_H$.

  But coordinate transformation contains singularity at $r = r_H$, whereas in Kruskal case no such singularity.

  $\Rightarrow$ Painleve coords. are not suitable in this analysis.

- Arbitrariness in $T_{in} = T_{out}$, $X_{in} = X_{out}$:

  $t_{in} = t_{out} + \frac{i\pi}{2\kappa}$, $r^*_{in} = r^*_{out} - \frac{i\pi}{2\kappa}$.

  Follow identical steps:

  Though $P^{(L)} = 1$, $P^{(R)} = e^{\frac{2\pi\omega}{\hbar\kappa}}$

  which diverges in the classical limit.

  $\Rightarrow$ Unphysical
  Exclude these set of relations.
  Of course, this second set is physically possible for the radiation from cosmic horizon (see *R.Banerjee and S.K.Modak*, **JHEP (2009)**).

Radiation spectrum

- Consider '*n*' no. of non-interacting virtual pairs created inside the horizon.
- Each of these pairs is represented by the modes: $\phi_{in}^{(R)}, \phi_{in}^{(L)}$.
- Physical state of the system:
  $$|\psi >= N\sum_n |n_{in}^{(L)} > \otimes |n_{in}^{(R)} >.$$

  $|n_{in}^{(L)} >$: corresponds to n no. of left modes and so on.
  N: Normalization const.

  The physical state of the system, observed from outside at infinity (**write the inside modes in terms of outside modes**):

  $$|\psi >= N\sum_n e^{-\frac{\pi n\omega}{\hbar\kappa}} |n_{out}^{(L)} > \otimes |n_{out}^{(R)} >.$$

- To determine N use normalization condition: $\langle\psi|\psi\rangle = 1$.

$$\Rightarrow N = \Big(\sum_n e^{-\frac{2\pi n\omega}{\hbar\kappa}}\Big)^{-\frac{1}{2}};$$

For bosons: $n = 0, 1, 2, ...$;
For fermions: $n = 0, 1$.

$$\Rightarrow N_{Bose} = \Big(1 - e^{-\frac{2\pi\omega}{\hbar\kappa}}\Big)^{\frac{1}{2}};$$

$$N_{Fermi} = \Big(1 + e^{-\frac{2\pi\omega}{\hbar\kappa}}\Big)^{-\frac{1}{2}}$$

Now the discussions will be on bosons; Similar steps follow for fermions.

- The physical state of the system as observed from infinity (for boson):

$$|\psi\rangle_{Bose} = \left(1 - e^{-\frac{2\pi\omega}{\hbar\kappa}}\right)^{\frac{1}{2}} \sum_n e^{-\frac{\pi n\omega}{\hbar\kappa}} |n^{(L)}_{out}\rangle \otimes |n^{(R)}_{out}\rangle$$

Density matrix:

$$\hat{\rho}_{Bose} = |\psi\rangle_{Bose}\langle\psi|_{Bose} =$$

$$\left(1 - e^{-\frac{2\pi\omega}{\hbar\kappa}}\right) \sum_{n,m} e^{-\frac{\pi n\omega}{\hbar\kappa}} e^{-\frac{\pi m\omega}{\hbar\kappa}} |n^{(L)}_{out}\rangle \otimes |n^{(R)}_{out}\rangle\langle m^{(R)}_{out}| \otimes \langle m^{(L)}_{out}|$$

- Trace out the L-modes $\Rightarrow$ The density matrix for R- modes:

$$\hat{\rho}^{(R)}_{Bose} = \left(1 - e^{-\frac{2\pi\omega}{\hbar\kappa}}\right) \sum_n e^{-\frac{2\pi n\omega}{\hbar\kappa}} |n^{(R)}_{out}\rangle\langle n^{(R)}_{out}|$$

- Average no. of particles detected at asymptotic infinity:

$$\langle n \rangle_{Bose} = \mathrm{trace}\Big(\hat{n}\hat{\rho}^{(R)}_{Bose}\Big)$$

$$= \Big(1 - e^{-\frac{2\pi\omega}{\hbar\kappa}}\Big)\sum_n n e^{-\frac{2\pi n\omega}{\hbar\kappa}}$$

$$\Rightarrow \langle n \rangle_{Bose} = \frac{1}{e^{\frac{2\pi\omega}{\hbar\kappa}} - 1}.$$

- Similar steps $\Rightarrow$

$$\langle n \rangle_{Fermi} = \frac{1}{e^{\frac{2\pi\omega}{\hbar\kappa}} + 1}.$$

- Perfect black body spectrum.

  Temp. $T_H = \frac{\hbar\kappa}{2\pi}$.

- Flux $= \frac{1}{\pi}\int_0^\infty \frac{\omega d\omega}{e^{\frac{2\pi\omega}{\hbar\kappa}} + 1} = \frac{\hbar^2\kappa^2}{48\pi}$

  This is a new result in the tunneling approach. Previously, only the temp. was obtained by "detailed balance" condition.

Comparison between two methods

Anomaly method		Tunneling method	
1.	Near horizon theory is 2-D, $(r-t)$ sector is important.	1.	Tunneling occurs radially, $(r-t)$ sector is necessary.
2.	Left modes lost inside the black hole. No contribution to Hawking flux from left mode.	2.	$P^L = 1 \Rightarrow$ left mode is trapped inside the black hole. Left mode gives no contribution to outgoing probability.

Both theories are chiral satisfying

$\Rightarrow$ Chirality condition:

$$J_\mu = \pm\sqrt{-g}\epsilon_{\mu\nu}J^\nu$$

$$T_{\mu\nu} = \pm\frac{1}{2}(\bar{\epsilon}_{\mu\sigma}T^\sigma_\nu + \bar{\epsilon}_{\nu\sigma}T^\sigma_\mu) + \frac{1}{2}g_{\mu\nu}T^\alpha_\alpha.$$

Anomaly method		Tunneling method	
3.	Gravitational anomaly from Chirality condition:	3.	Modes are obtained from chirality condition.

Anomaly expression is actually derived from these modes.

4.	Solve anomaly equation Use covariant B.C. Asymptotic infinity $\Rightarrow$ Hawking flux	4.	Calculate reduced density matrix. Calculate average no of R particles. Integrate. Hawking flux corresponding to Hawking temp.

Conclusions

- The notion of chirality pervades the anomaly and tunneling methods thereby providing a close connection between them.
- Using the same chirality condition Hawking flux in anomaly method and in tunneling method is computed.

Black hole spectroscopy

- D. Christodoulou, **PRL (1970)**: Black hole horizon area behaves as a classical adiabatic invariant quantity.
- J. D. Bekenstein, **PRD (1973)**: *Ehrenfest principle*: this classical quantity corresponds to quantum entity with discrete spectrum; so black hole area should be quantized.

  $A_n = 8\pi l_p^2 n$; n is an large integar.

OUR WORK

We shall discuss area/entropy quantization through tunneling mechanism.

Based on: Uncertainty principle and entropy as the loss of information.

- Recall: The modes of a pair created inside the black hole as seen from outside are:

$$\phi^{(R)}_{in} = e^{-\frac{\pi\omega}{\hbar\kappa}}\phi^{(R)}_{out};$$

$$\phi^{(L)}_{in} = \phi^{(L)}_{out}.$$

Also, the L-mode is trapped inside the black hole and ω is the effective energy of the emitted particle.

- The average value of ω:

$$<\omega>= \frac{\int_0^\infty \left(\phi^{(R)}_{in}\right)^* \omega\phi^{(R)}_{in} d\omega}{\int_0^\infty \left(\phi^{(R)}_{in}\right)^* \phi^{(R)}_{in} d\omega} =$$

$$\frac{\int_0^\infty e^{-\frac{\pi\omega}{\hbar\kappa}} \left(\phi^{(R)}_{out}\right)^* \omega e^{-\frac{\pi\omega}{\hbar\kappa}}\phi^{(R)}_{out} d\omega}{\int_0^\infty e^{-\frac{\pi\omega}{\hbar\kappa}} \left(\phi^{(R)}_{out}\right)^* e^{-\frac{\pi\omega}{\hbar\kappa}}\phi^{(R)}_{out} d\omega} = T_H;\ T_H = \frac{\hbar\kappa}{2\pi}.$$

- Similarly,

$$<\omega^2>=\frac{\int_0^\infty e^{-\frac{\pi\omega}{\hbar\kappa}}\left(\phi^{(R)}_{out}\right)^*\omega^2 e^{-\frac{\pi\omega}{\hbar\kappa}}\phi^{(R)}_{out}d\omega}{\int_0^\infty e^{-\frac{\pi\omega}{\hbar\kappa}}\left(\phi^{(R)}_{out}\right)^* e^{-\frac{\pi\omega}{\hbar\kappa}}\phi^{(R)}_{out}d\omega}=2T_H^2.$$

- Uncertaintity in the detected energy:

$$(\Delta\omega)=\sqrt{<\omega^2> - <\omega>^2}=T_H.$$

Comments: 1. Heisenberg uncertainty relation : $(\Delta\omega)(\Delta\tau)\geq\hbar\Rightarrow(\Delta\tau)\geq\frac{\hbar}{T_H}=\frac{2\pi}{\kappa}$.

Period of Euclidean time of Hawking and Gibbons.

2. Kastrup (*PLB 385, 75 (1996)*) obtained the mass spectrum of a black hole by postulating the above periodic boundary condition of time.

- Information theory: entropy is the lack of information.

  Uncertainty can be seen as the minimum lack of information in energy of the emitted particle.

  Use first law of black hole mechanics:

  $T_H(\Delta S_{bh})_{min} = (\Delta\omega)_{min}$.

  $\Rightarrow (\Delta S_{bh})_{min} = 1$.

  1. Entropy is quantized in units of identity.
  2. Entropy spectrum is equidistant.

- Einstein gravity: $S_{bh} = \frac{A}{4l_{pl}^2}$.

  $\Rightarrow (\Delta A)_{min} = 4l_{pl}^2$.

  $\Rightarrow$ Area spectrum : $A_n = 4l_{pl}^2 n$; Area spectrum is equally spaced.

- Of course, if the analysis was done for a more general theory (e.g. *Einstein-Gauss-Bonnet gravity*), the spectrum of entropy is equidistant where as that of area is not because in such case the entropy is not proportional to area.

- For more general theory, the notion of the quantum of entropy is more natural than the quantum of area.

- Since our analysis is semiclassical, the area/entropy spectrum is valid only for large n.

- The quantum of area obtained here ($A_n = 4l_p^2 n$) is not equal to that obtained by Bekenstein ($A_n = 8\pi l_p^2 n$), but agrees with S. Hod **PRD (1999)**.

FINAL REMARKS

- Both the gravitational anomaly expression and the HJ equation are obtained from the chiral conditions.
- The Hawking flux has been calculated from anomaly expression as well as HJ equation. Here we reformulated the usual tunneling approach. The advantage is that the perfect black body spectrum is obtained instead of temp. only.
- A comparison between these two methods reveals that chirality provides a close connection between them.
- Finally, using the uncertainty principle and the concept of information theory (entropy as the loss of information) the entropy/area spectrum has been discussed.
- The advantage of this method is that it does not require any adiabatic invariant quantity.

Thank You