Chapter 7

Pull back and push forward

Two important concepts are those of pull back (or pull-back or pull-back) and push forward (or push-forward or pushforward) of maps between manifolds.

- Given manifolds $M_1, M_2, M_3$ and maps $f : M_1 \to M_2, g : M_2 \to M_3$, the pullback of $g$ under $f$ is the map $f^* g : M_1 \to M_3$ defined by
  \[ f^* g = g \circ f. \]  
  \(7.1\)

So in particular, if $M_1$ and $M_2$ are two manifolds with a map $f : M_1 \to M_2$ and $g : M_2 \to \mathbb{R}$ is a function on $M_2$, the pullback of $g$ under $f$ is a function on $M_1$,

\[ f^* g = g \circ f. \]  
\(7.2\)

While this looks utterly trivial at this point, this concept will become increasingly useful later on.

- Given two manifolds $M_1$ and $M_2$ with a smooth map $f : M_1 \to M_2, P \mapsto Q$ the pushforward of a vector $v \in T_P M_1$ is a vector $f_* v \in T_Q M_2$ defined by
  \[ f_* v = v \circ f \]  
  \(7.3\)

for all smooth functions $g : M_2 \to \mathbb{R}$.

Thus we can write

\[ f_* v(g) = v(f^* g). \]  
\(7.4\)
The pushforward is linear,
\[ f_*(v_1 + v_2) = f_*v_1 + f_*v_2 \] (7.5)
\[ f_*(\lambda v) = \lambda f_*v. \] (7.6)

And if \( M_1, M_2, M_3 \) are manifolds with maps \( f : M_1 \to M_2, g : M_2 \to M_3 \), it follows that
\[ (g \circ f)_* = g_*f_*, \quad \text{i.e.} \]
\[ (g \circ f)_*v = g_*f_*v \quad \forall v \in T_P M_1. \] (7.7)

Remember that we can think of a vector \( v \) as an equivalence class of curves \([\gamma]\). The pushforward of an equivalence class of curves is
\[ f_*v = f_*[\gamma] = [f \circ \gamma] \] (7.8)

Note that for this pushforward to be defined, we do not need the original maps to be 1-1 or onto. In particular, the two manifolds may have different dimensions.

Suppose \( M_1 \) and \( M_2 \) are two manifolds with dimension \( m \) and \( n \) respectively. So in the respective tangent spaces \( T_P M_1 \) and \( T_Q M_2 \) are also of dimension \( m \) and \( n \) respectively. So for a map \( f : M_1 \to M_2, P \mapsto Q \), the pushforward \( f_* \) will not have an inverse if \( m \neq n \).

Let us find the components of the pushforward \( f_*v \) in terms of the components of \( v \) for any vector \( v \). Let us in fact consider, given charts \( \varphi : P \mapsto (x^1, \ldots, x^m), \psi : Q \mapsto (y^1, \ldots, y^n) \) the pushforward of the basis vectors.

For the basis vector \( \left( \frac{\partial}{\partial x^i} \right)_P \), we want the pushforward \( f_* \left( \frac{\partial}{\partial x^i} \right)_P \), which is a vector in \( T_Q M_2 \), so we can expand it in the basis \( \left( \frac{\partial}{\partial y^\mu} \right)_Q \),
\[ f_* \left( \frac{\partial}{\partial x^i} \right)_P = \left( f_* \left( \frac{\partial}{\partial x^i} \right)_P \right)_\mu \left( \frac{\partial}{\partial y^\mu} \right)_Q \] (7.9)

In any coordinate basis, the components of a vector are given by the action of the vector on the coordinates as in Chap. 4,
\[ v^\mu_P = v_P (y^\mu) \] (7.10)

Thus we can write
\[ \left( f_* \left( \frac{\partial}{\partial x^i} \right)_P \right)_\mu = f_* \left( \frac{\partial}{\partial x^i} \right)_P (y^\mu) \] (7.11)
But
\[ f_*(v) = v(g \circ f), \quad (7.12) \]
so
\[ f_*(\partial_{x^i})_P (y^\mu) = \left( \frac{\partial y^\mu}{\partial x^i} \right)_P (y^\mu \circ f). \quad (7.13) \]
But \( y^\mu \circ f \) are the coordinate functions of the map \( f \), i.e., coordinates around the point \( f(P) = Q \). So we can write \( y^\mu \circ f \) as \( y^\mu(x) \), which is what we understand by this. Thus
\[
\left( f_*(\frac{\partial}{\partial x^i})_P \right)^\mu = \left( \frac{\partial}{\partial x^j} \right)_P (y^\mu \circ f) = \frac{\partial y^\mu(x)}{\partial x^i} \bigg|_P. \quad (7.14)
\]
Because we are talking about derivatives of coordinates, these are actually done in charts around \( P \) and \( Q = f(P) \), so the chart maps are hidden in this equation.

- The right hand side is called the **Jacobian matrix** (of \( y^\mu(x) = y^\mu \circ f \) with respect to \( x^i \)). Note that since \( m \) and \( n \) may be unequal, this matrix need not be invertible and a determinant may not be defined for it.

For the basis vectors, we can then write
\[
f_*(\frac{\partial}{\partial x^i})_P = \frac{\partial y^\mu(x)}{\partial x^i} \bigg|_P \left( \frac{\partial}{\partial y^\mu} \right)_P f(P). \quad (7.15)
\]
Since \( f_* \) is linear, we can use this to find the components of \((f_*v)_Q\) for any vector \( v_P \),
\[
f_*v_P = f_* \left[ v^i_P \left( \frac{\partial}{\partial x^i} \right)_P \right] = v^i_P f_* \left( \frac{\partial}{\partial x^i} \right)_P = v^i_P \frac{\partial y^\mu(x)}{\partial x^i} \bigg|_P \left( \frac{\partial}{\partial y^\mu} \right)_P f(P) \quad (7.16)
\]
\[
\Rightarrow \quad (f_*v_P)^\mu = v^i_P \frac{\partial y^\mu(x)}{\partial x^i} \bigg|_P. \quad (7.17)
\]
Note that since \( f_* \) is linear, we know that the components of \( f_*v \) should be linear combinations of the components of \( v \), so we can
already guess that \((f_* v_p)^\mu = A^\mu_i v^i_p\), for some matrix \(A^\mu_i\). The matrix is made of first derivatives because vectors are first derivatives.

Another example of the pushforward map is the following. Remember that tangent vectors are derivatives along curves. Suppose \(v_p \in T_p \mathcal{M}\) is the derivative along \(\gamma\). Since \(\gamma : I \to \mathcal{M}\) is a map, we can consider pushforwards under \(\gamma\), of derivatives on \(I\). Thus for \(\gamma : I \to \mathcal{M}, t \mapsto \gamma(t) = P\), and for some \(g : \mathcal{M} \to \mathbb{R}\),

\[
\gamma_* \left( \frac{d}{dt} \right)_{t=0} g = \frac{d}{dt} (g \circ \gamma)|_{t=0} = \dot{\gamma}_P (g)|_{t=0} = v_P (g), \tag{7.18}
\]

so

\[
\gamma_* \left( \frac{d}{dt} \right)_{t=0} = v_P. \tag{7.19}
\]

- We can use this to give another definition of integral curves. Suppose we have a vector field \(v\) on \(\mathcal{M}\). Then the integral curve of \(v\) passing through \(P \in \mathcal{M}\) is a curve \(\gamma : t \mapsto \gamma(t)\) such that \(\gamma(0) = P\) and

\[
\gamma_* \left( \frac{d}{dt} \right)_{t} = v|_{\gamma(t)} \tag{7.20}
\]

for all \(t\) in some interval containing \(P\). □

Even though in order to define the pushforward of a vector \(v\) under a map \(f\), we do not need \(f\) to be invertible, the pushforward of a vector field can be defined only if \(f\) is both one-to-one and onto.

If \(f\) is not one-to-one, different points \(P\) and \(P'\) may have the same image, \(f(P) = Q = f(P')\). Then for the same vector field \(v\) we must have

\[
f_* v|_Q = f_* (v_p) = f_* (v_{p'}) , \tag{7.21}
\]

which may not be true. And if \(f : \mathcal{M} \to \mathcal{N}\) is not onto, \(f_* v\) will be meaningless outside some region \(f(\mathcal{M})\), so \(f_* v\) will not be a vector field on \(\mathcal{N}\).

If \(f\) is one-to-one and onto, it is a diffeomorphism, in which case vector fields can be pushed forward, by the rule

\[
(f_* v)_{f(P)} = f_* (v_p) . \tag{7.22}
\]