Chapter 25

Curvature

We start with a connection D, two vector fields v, w on B, and a section s, all on some associated vector bundle of some principal G-bundle E. Then D_v, D_w are both maps $\Gamma(E) \to \Gamma(E)$.

We will define the curvature of this connection D as a rule F which, given two vector fields v, w, produces a linear map $F(v,w) : \Gamma(E) \to \Gamma(E)$ by

$$F(v,w)s = D_v D_w s - D_w D_v s - D_{[v,w]} s\,. \tag{25.1}$$

Remember that

$$\begin{aligned} D_v s &= D_{v^\mu \partial_\mu} s = v^\mu D_\mu s \\ &= v^\mu \left(\partial_\mu s^i + A_\mu{}^i{}_j\, s^j \right) e_i \\ &= v(s^i) e_i + v^\mu A_\mu{}^i{}_j\, s^j e_i\,. \end{aligned} \tag{25.2}$$

$D_\mu s$ is again a section. So we can act with D on it and write

$$\begin{aligned} D_v D_w s &= D_v \left[w(s^j) e_j + \left(w^\nu A_\nu{}^j{}_k\, s^k \right) e_j \right] \\ &= v\left(w\left(s^j \right) \right) e_j + v \left(w^\nu A_\nu{}^j{}_k\, s^k \right) e_j \\ &\quad + \left(w\left(s^j \right) + w^\nu A_\nu{}^j{}_k\, s^k \right) v^\mu A_\mu{}^i{}_j\, s^j e_i\,. \end{aligned} \tag{25.3}$$

Since the connection components $A_\nu{}^j{}_k$ are functions, we can write

$$v \left(w^\nu A_\nu{}^j{}_k\, s^k \right) = v(w^\nu) A_\nu{}^j{}_k\, s^k + w^\nu v(A_\nu{}^j{}_k) s^k + w^\nu A_\nu{}^j{}_k v(s^k)\,. \tag{25.4}$$

Inserting this into the previous equation and writing $D_w D_v s$ similarly, we find

$$\begin{aligned} D_v D_w s - D_w D_v s = [v\,,w](s^i)e_i + [v\,,w]^\mu A_\mu{}^i{}_j\, s^j e_i \\ + \left(w^\nu v\left(A_\nu{}^i{}_j\right) - v^\mu w\left(A_\mu{}^i{}_j\right)\right) s^j e_i \\ + v^\mu w^\nu \left(A_\mu{}^i{}_j A_\nu{}^j{}_k - A_\nu{}^i{}_j A_\mu{}^j{}_k\right) s^k e_i\,. \end{aligned} \tag{25.5}$$

Also,

$$D_{[v\,,w]} s = [v\,,w](s^i)e_i + [v\,,w]^\mu A_\mu{}^i{}_j\, s^j e_i\,, \tag{25.6}$$

so that

$$F(v,w)s = v^\mu w^\nu \left(\partial_\mu A_\nu{}^i{}_j - \partial_\nu A_\mu{}^i{}_j + A_\mu{}^i{}_k A_\nu{}^k{}_j - A_\nu{}^i{}_k A_\mu{}^k{}_j\right) s^j e_i\,. \tag{25.7}$$

Thus we can define $F_{\mu\nu}$ by

$$F(\partial_\mu, \partial_\nu)s = F_{\mu\nu}s = (F_{\mu\nu}s)^i\, e_i = (F_{\mu\nu})^i{}_j\, s^j e_i\,, \tag{25.8}$$

so that

$$(F_{\mu\nu})^i{}_j = \partial_\mu A_\nu{}^i{}_j - \partial_\nu A_\mu{}^i{}_j + A_\mu{}^i{}_k A_\nu{}^k{}_j - A_\nu{}^i{}_k A_\mu{}^k{}_j\,. \tag{25.9}$$

Note that since coordinate basis vector fields commute, $[\partial_\mu\,, \partial_\nu] = 0$,

$$F_{\mu\nu}s = F(\partial_\mu, \partial_\nu)s = D_\mu D_\nu s - D_\nu D_\mu s = [D_\mu\,, D_\nu]s\,. \tag{25.10}$$

- It is not very difficult to work out that the curvature acts linearly on the module of sections,

$$F(u,v)(s_1 + f s_2) = F(u,v)s_1 + fF(u,v)s_2\,, \tag{25.11}$$

where $f \in C^\infty(B)$. Also,

$$F(u, v + fw)\, s = F(u,v)s + fF(u,w)s\,. \tag{25.12}$$

- For coordinate basis vector fields $[\partial_\mu\,, \partial_\nu] = 0$, so

$$F_{\mu\nu}s = F(\partial_\mu, \partial_\nu)s = D_\mu D_\nu s - D_\nu D_\mu s = [D_\mu\,, D_\nu]s\,. \tag{25.13}$$

Since $F(\partial_\mu, \partial_\nu)s$ is a section, so is

$$D_\lambda\left(F_{\mu\nu}s\right) = D_\lambda [D_\mu\,, D_\nu]s\,. \tag{25.14}$$

Similarly, since $D_\lambda s$ is a section, so is

$$F_{\mu\nu} D_\lambda s = [D_\mu \,, D_\nu] D_\lambda s \,. \tag{25.15}$$

Thus

$$D_\lambda \left(F_{\mu\nu} s\right) - F_{\mu\nu} D_\lambda s = [D_\lambda \,, [D_\mu \,, D_\nu]] s \,. \tag{25.16}$$

Considering C^∞ sections, and noting that maps are associative under map composition, we find that

$$[D_\lambda \,, [D_\mu \,, D_\nu]] s + \text{cyclic} = 0 \,. \tag{25.17}$$

On the other hand,

$$F_{\mu\nu} s = \left(F_{\mu\nu} s\right)^i e_i = \left(F_{\mu\nu}{}^i{}_j s^j\right) e_i \,, \tag{25.18}$$

where $F_{\mu\nu}{}^i{}_j$ and s^i are in $C^\infty(B)$. So we can write

$$\begin{aligned} D_\lambda \left(F_{\mu\nu} s\right) &= \partial_\lambda \left(F_{\mu\nu}{}^i{}_j s^j\right) + \left(F_{\mu\nu}{}^i{}_j s^j\right) D_\lambda e_i \\ &= \left(\partial_\lambda F_{\mu\nu}{}^i{}_j\right) s^j e_i + F_{\mu\nu}{}^i{}_j \left(\partial_\lambda s^j\right) e_i + F_{\mu\nu}{}^i{}_j s^j A_\lambda{}^k{}_i e_k \\ &= \left(\partial_\lambda F_{\mu\nu}{}^i{}_j + F_{\mu\nu}{}^k{}_j A_\lambda{}^i{}_k\right) s^j e_i + F_{\mu\nu}{}^i{}_j \left(\partial_\lambda s^j\right) e_i \\ &\qquad - F_{\mu\nu}{}^i{}_k A_\lambda{}^k{}_j s^j e_i + F_{\mu\nu}{}^i{}_j A_\lambda{}^j{}_k s^k e_i \\ &= \left(D_\lambda F_{\mu\nu}\right)^i{}_j s^j e_i + F_{\mu\nu}{}^i{}_j \left(D_\lambda s\right)^j e_i \,, \end{aligned} \tag{25.19}$$

where we have defined $(D_\lambda F_{\mu\nu})$ by $\left(D_\lambda F_{\mu\nu}\right)^i{}_j$ in this. Then this is a Leibniz rule,

$$D_\lambda \left(F_{\mu\nu} s\right) = \left(D_\lambda F_{\mu\nu}\right) s + F_{\mu\nu} \left(D_\lambda s\right) \,. \tag{25.20}$$

Then we can write

$$\begin{aligned} & D_\lambda \left(F_{\mu\nu} s\right) - F_{\mu\nu} \left(D_\lambda s\right) + \text{cyclic} = 0 \\ \Rightarrow \quad & \left(D_\lambda F_{\mu\nu}\right) s + \text{cyclic} = 0 \qquad \forall \text{s} \\ \Rightarrow \quad & D_\lambda F_{\mu\nu} + \text{cyclic} = 0 \,. \end{aligned} \tag{25.21}$$

This is known as the **Bianchi identity** . □

Given D and g such that $g(x) \in G$, we have D' given by

$$D'_v \phi = g D_v \left(g^{-1} \phi\right) \,. \tag{25.22}$$

Then

$$D'_u D'_v \phi = D'_u \left(g D_v \left(g^{-1} \phi \right) \right) = g D_u D_v \left(g^{-1} \phi \right) \,, \tag{25.23}$$

and thus

$$\begin{aligned} F'(u,v)\,\phi &\equiv \left(D'_u D'_v - D'_v D'_u - D'_{[u\,,v]} \right) \phi \\ &= g D_u D_v \left(g^{-1} \phi \right) - D_v D_u \left(g^{-1} \phi \right) - g D_{[u\,,v]} \left(g^{-1} \phi \right) \\ &= g F(u,v) g^{-1} \phi \\ \Rightarrow \qquad F'_{\mu\nu} &= g \circ F_{\mu\nu} \circ g^{-1} \,. \end{aligned} \tag{25.24}$$

As before, g is in some representation of G, and D (and thus F) acts on the same representation. This is the meaning of the statement that the curvature is ***gauge covariant***. □