Chapter 23

Fiber bundles

Consider a manifold $\mathcal{M}$ with the tangent bundle $T\mathcal{M} = \bigcup_{P\in\mathcal{M}} T_P\mathcal{M}$. Let us look at this more closely. $T\mathcal{M}$ can be thought of as the original manifold $\mathcal{M}$ with a tangent space stuck at each point $P \in \mathcal{M}$. Thus there is a projection map $\pi : T\mathcal{M} \to \mathcal{M}, \quad T_P\mathcal{M} \mapsto P$, which associates the point $P \in \mathcal{M}$ with $T_P\mathcal{M}$.

Then we can say that $T\mathcal{M}$ consists of pooints $P \in \mathcal{M}$ and vectors $v \in T_P\mathcal{M}$ as an ordered pair (P, v_P). Then in the neighbourhood of any point P, we can think of $T\mathcal{M}$ as a **product manifold**, i.e. as the set of ordered pairs (P, v_P).

This is generalized to the definition of a **fiber bundle**. Locally a fiber bundle is a product manifold $E = B \times F$ with the following properties.

• B is a manifold called the **base manifold**, and F is another manifold called the **typical fiber** or the **standard fiber**.

• There is a projection map $\pi : E \to B$, and if $P \in B$, the pre-image $\pi^{-1}(P)$ is homeomorphic, i.e. bicontinuously isomorphic, to the standard fiber. □

E is called the **total space**, but usually it is also called the bundle, even though the bundle is actually the triple (E, π, B).

• E is locally a product space. We express this in the following way. Given an open set U_i of B, the pre-image $\pi^{-1}(U_i)$ is homeomorphic to $U_i \times F$, or in other words there is a bicontinuous isomorphism $\varphi_i : \pi^{-1}(U_i) \to U_i \times F$. The set $\{U_i, \varphi_i\}$ is called a **local trivialization** of the bundle. □

• If E can be written globally as a product space, i.e. $E = B \times F$, it is called a **trivial bundle**. □

• This description includes a homeomorphism $\pi^{-1}(P) \to F$ for each $P \in U_i$. Let us denote this map by $h_i(P)$. Then in some overlap $U_i \cap U_j$ the fiber on $P, \pi^{-1}(P)$, has homeomorphisms $h_i(P)$ and $h_j(P)$ onto F. It follows that $h_j(P) \cdot h_i(P)^{-1}$ is a homeomorphism $F \to F$. These are called **transition functions**. The transition functions $F \to F$ form a group, called the **structure group** of F. □

Let us consider an example. Suppose $B = S^1$. Then the tangent bundle $E = TS^1$ has $F = \mathbb{R}$ and $\pi(P, v) \mapsto P$, where $P \in S^1, v \in TS^1$. Consider a covering of S^1 by open sets U_i, and let the coordinates of $U_i \subset S^1$ be denoted by λ_i. Then any vector at $T_P S^1$ can be written as $v = a_i \dfrac{d}{d\lambda_i}$ (no sum) for $P \in U_i$.

So we can define a homeomorphism $h_i(P) : T_P S^1 \to \mathbb{R}, v \mapsto a_i$ (fixed i). If $P \in U_i \cap U_j$ there are two such homeomorphisms $TS^1 \to \mathbb{R}$, and since λ_i and λ_j are independent, a_i and a_j are also independent.

Then $h_i(P) \cdot h_j(P)^{-1} : F \to F$ (or $\mathbb{R} \to \mathbb{R}$) maps a_j to a_i. The homeomorphism, which in this case relates the component of the vector in two coordinate systems, is simply multiplication by the number $r_{ij} = \dfrac{a_i}{a_j} \in \mathbb{R}\backslash\{0\}$. So the structure group is $\mathbb{R}\backslash\{0\}$ with multiplication.

For an n-dimensional manifold $\mathcal{M}$, the structure group of $T\mathcal{M}$ is $GL(n, \mathbb{R})$.

• A fiber bundle where the standard fiber is a vector space is called a **vector bundle**. □

A cylinder can be made by glueing two opposite edges of a flat strip of paper. This is then a Cartesian product of acircle S^1 with a line segment I. So $B = S^1, F = I$ and this is a trivial bundle, i.e. globally a product space. On the other hand, a Möbius strip is obtained by twisting the strip and then glueing. Locally for some open set $U \subsetneq S^1$ we can still write a segment of the Möbius strip as $U \times I$, but the total space is no longer a product space. As a bundle, the Möbius strip is non-trivial.

• Given two bundles (E_1, π_1, B_1) and (E_2, π_2, B_2), the relevant or useful maps between these are those which preserve the bundle structure locally, i.e. those which map fibers into fibers. They are called **bundle morphisms**. □

A bundle morphism is a pair of maps $(F, f), F : E_1 \to E_2, f :$

$B_1 \to B_2$, such that $\pi_2 \circ F = f \circ \pi_1$. (This is of course better understood in terms of a commutative diagram.)

Not all systems of coordinates are appropriate for a bundle. But it is possible to define a set of **fiber coordinates** in the following way. Given a differentiable fiber bundle with n-dimensional base manifold B and p-dimensional fiber F, the coordinates of the bundle are given by bundle morphisms onto open sets of $\mathbb{R}^n \times \mathbb{R}^p$. □

• Given a manifold $\mathcal{M}$ the tangent space $T_P\mathcal{M}$, consider $A_P = (e_1, \cdots, e_n)$, a set of n linearly independent vectors at P. A_P is a basis in $T_P\mathcal{M}$. The typical fiber in the **frame bundle** is the set of all bases, $F = \{A_P\}$. □

Given a particular basis $\overline{A}_P = (\bar{e}_1, \cdots, \bar{e}_n)$, any basis A_P may be expressed as

$$e_i = a_i^j \bar{e}_j \,. \tag{23.1}$$

The numbers a_i^j can be thought of as the components of a matrix, which must be invertible so that we can recover the original basis from the new one. Thus, starting from any one basis, any other basis can be reached by an $n \times n$ invertible matrix, and any $n \times n$ invertible matrix produces a new basis. So there is a bijection between the set of all frames in $T_P\mathcal{M}$ and $GL(n, \mathbb{R})$.

Clearly the structure group of the typical fiber of the frame bundle is also $GL(n, \mathbb{R})$.

• A fiber bundle in which the typical fiber F is identical (or homeomorphic) to the structure group G, and G acts on F by left translation is called a **principal fiber bundle**. □

Example: 1. Typical fiber $= S^1$, structure group $U(1)$.

2. Typical fiber $= S^3$, structure group $SU(2)$.

3. Bundle of frames, for which the typical fiber is $GL(n, \mathbb{R})$, as is the structure group.

• A **section** of a fiber bundle (E, π, B) is a mapping $s : B \to E, p \mapsto s(p)$, where $p \in B, s(p) \in \pi^{-1}(p)$. So we can also say $\pi \circ s$ = identity. □

Example: A vector field is a section of the tangent bundle, $v : P \mapsto v_P$.

Example: A function on $\mathcal{M}$ is a section of the bundle which locally looks like $\mathcal{M} \times \mathbb{R}$ (or $\mathcal{M} \times \mathbb{C}$ if we are talking about complex functions).

• Starting from the tangent bundle we can define the **cotangent bundle**, in which the typical fiber is the dual space of the tangent space. This is written as $T^*\mathcal{M}$. As we have seen before, a section of $T^*\mathcal{M}$ is a 1-form field on $\mathcal{M}$. □

• Remember that a **vector bundle** $F \to E \xrightarrow{\pi} B$ is a bundle in which the typical fiber F is a vector space. □

• A vector bundle $(E, \widetilde{\pi}, B, F, G)$ with typical fiber F and structure group G is said to be **associated** to the principal bundle (P, π, B, G) by the representation $\{D(g)\}$ of G on F if its transition functions are the images under D of the transition functions of P . □

That is, suppose we have a covering $\{U_i\}$ of B , and local trivialization of P with respect to this covering is $\Phi_i : \pi^{-1}(U_i) \to U_i \times G$, which is essentially the same as writing $\Phi_{i,x} : \pi^{-1}(x) \to G\,,\ x \in U_i$. Then the transition functions of P are of the form

$$g_{ij} = \Phi_i \circ {\Phi_j}^{-1} : U_i \cap U_j \to G\,. \tag{23.2}$$

The transition functions of E corresponding to the same covering of B are given by $\phi_i : \widetilde{\pi}^{-1}(U_i) \to U_i \times F$ with $\phi_i \circ {\phi_j}^{-1} = D(g_{ij})$. That is, if v_i and v_j are images of the same vector $v_x \in F_x$ under overlapping trivializations ϕ_i and ϕ_j , we must have

$$v_i = D\left(g_{ij}(x)\right) v_j\,. \tag{23.3}$$

A more physical way of saying this is that if two observers look at the same vector at the same point, their observations are relatted by a group transformation $(p, v) \simeq (p, D(g_{ij}v)$.

• These relations are what are called **gauge transformations** in physics, and G is called the **gauge group**. Usually G is a Lie group for reasons of continuity. □

Fields appearing in various physical theories are sections of vector bundles, which in some trivialization look like $U_\alpha \times V$ where U_α is some open neighborhood of the point we are interested in, and V is a vector space. V carries a representation of some group G , usually a Lie group, which characterizes the theory.

To discuss this a little more concretely, let us consider an associated vector bundle $(E, \widetilde{\pi}, B, F, G)$ of a principal bundle (P, π, B, G). Then the transition functions are in some representation of the group G . Because the fiber carries a representation $\{D(g)\}$ of G , there are always linear transformations $T_x : E_x \to E_x$ which are members of the representation $\{D(g)\}$. Let us write the space of all sections of this bundle as $\Gamma(E)$. An element of $\Gamma(E)$ is a map from the base spacce to the bundle. Such a map assigns an element of V to each point of the base space.

• We say that a linear map $T : \Gamma(E) \to \Gamma(E)$ is a **gauge trans formation** if at each point x of the base space, $T_x \in \{D(g)\}$ for some g, i.e. if

$$T_x : (x, v)_\alpha \mapsto (x, D(g)v)_\alpha\,, \tag{23.4}$$

for some $g \in G$ and for $(x, v)_\alpha \in U_\alpha \times F$. In other words, a gauge transformation is a representation-valued linear transformation of the sections at each point of the base space. The right hand side is often written as $(x, gv)_\alpha$. □

This definition is independent of the choice of U_α. To see this, consider a point $x \in U_\alpha \cap U_\beta$. Then

$$(x, v)_\alpha = (x, g_{\beta\alpha} v)_\beta\,. \tag{23.5}$$

In the other notation we have been using, v_α and v_β are images of the same vector $v_x \in V_x$, and $v_\beta = D(g_{\beta\alpha})v_\alpha$. A gauge transformation T acts as

$$T_x : (x, v)_\alpha \mapsto (x, gv)_\alpha\,. \tag{23.6}$$

But we also have

$$(x, gv)_\alpha = (x, g_{\beta\alpha} gv)_\beta \tag{23.7}$$

using Eq. (23.5). So it is also true that

$$T_x : (x, g_{\beta\alpha} v)_\beta \mapsto (x, g_{\beta\alpha} gv)_\beta\,. \tag{23.8}$$

Since F carries a representation of G, we can think of gv as a change of variables, i.e. define $v' = g_{\beta\alpha} v$. Then Eq. (23.8) can be written also as

$$T_x : (x, v')_\beta \mapsto (x, g'v')_\beta\,, \tag{23.9}$$

where now $g' = g_{\beta\alpha} g {g_{\beta\alpha}}^{-1}$. So T is a gauge transformation in U_β as well. The definition of a gauge transformation is independent of the choice of U_α , but T itself is not. The set of all gauge transformations $\mathscr{G}$ is a group, with

$$(gh)(x) = g(x)h(x)\,, \quad (g^{-1})(x) = (g(x))^{-1}\,. \tag{23.10}$$

• The groups G and $\mathscr{G}$ arre both called the **gauge group** by different people. □