Chapter 23

Fiber bundles

Consider a manifold $\mathcal{M}$ with the tangent bundle $T\mathcal{M} = \bigcup_{P \in \mathcal{M}} T_P \mathcal{M}$. Let us look at this more closely. $T\mathcal{M}$ can be thought of as the original manifold $\mathcal{M}$ with a tangent space stuck at each point $P \in \mathcal{M}$. Thus there is a projection map $\pi : T\mathcal{M} \to \mathcal{M}$, $T_P \mathcal{M} \mapsto P$, which associates the point $P \in \mathcal{M}$ with $T_P \mathcal{M}$.

Then we can say that $T\mathcal{M}$ consists of points $P \in \mathcal{M}$ and vectors $v \in T_P \mathcal{M}$ as an ordered pair $(P, v_P)$. Then in the neighbourhood of any point $P$, we can think of $T\mathcal{M}$ as a product manifold, i.e. as the set of ordered pairs $(P, v_P)$.

This is generalized to the definition of a fiber bundle. Locally a fiber bundle is a product manifold $E = B \times F$ with the following properties.

- $B$ is a manifold called the base manifold, and $F$ is another manifold called the typical fiber or the standard fiber.
- There is a projection map $\pi : E \to B$ and if $P \in B$, the pre-image $\pi^{-1}(P)$ is homeomorphic, i.e. bicontinuously isomorphic, to the standard fiber.

$E$ is called the total space, but usually it is also called the bundle, even though the bundle is actually the triple $(E, \pi, B)$.

- $E$ is locally a product space. We express this in the following way. Given an open set $U_i$ of $B$, the pre-image $\pi^{-1}(U_i)$ is homeomorphic to $U_i \times F$, or in other words there is a bicontinuous isomorphism $\varphi_i : \pi^{-1}(U_i) \to U_i \times F$. The set $\{U_i, \varphi_i\}$ is called a local trivialization of the bundle.
- If $E$ can be written globally as a product space, i.e. $E = B \times F$, it is called a trivial bundle.
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• This description includes a homeomorphism \( \pi^{-1}(P) \rightarrow F \) for each \( P \in U_i \). Let us denote this map by \( h_i(P) \). Then in some overlap \( U_i \cap U_j \) the fiber on \( P, \pi^{-1}(P) \), has homeomorphisms \( h_i(P) \) and \( h_j(P) \) onto \( F \). It follows that \( h_j(P) \cdot h_i(P)^{-1} \) is a homeomorphism \( F \rightarrow F \). These are called transition functions. The transition functions \( F \rightarrow F \) form a group, called the structure group of \( F \). □

Let us consider an example. Suppose \( B = S^1 \). Then the tangent bundle \( E = TS^1 \) has \( F = \mathbb{R} \) and \( \pi(P,v) \rightarrow P \), where \( P \in S^1, v \in TS^1 \). Consider a covering of \( S^1 \) by open sets \( U_i \), and let the coordinates of \( U_i \subset S^1 \) be denoted by \( \lambda_i \). Then any vector at \( T_P S^1 \) can be written as \( v = a_i \frac{d}{d\lambda_i} \) (no sum) for \( P \in U_i \).

So we can define a homeomorphism \( h_i(P) : T_P S^1 \rightarrow \mathbb{R}, v \mapsto a_i \) (fixed \( i \)). If \( P \in U_i \cap U_j \) there are two such homeomorphisms \( TS^1 \rightarrow \mathbb{R} \), and since \( \lambda_i \) and \( \lambda_j \) are independent, \( a_i \) and \( a_j \) are also independent.

Then \( h_i(P) \cdot h_j(P)^{-1} : F \rightarrow F \) (or \( \mathbb{R} \rightarrow \mathbb{R} \)) maps \( a_j \) to \( a_i \). The homeomorphism, which in this case relates the component of the vector in two coordinate systems, is simply multiplication by the number \( r_{ij} = \frac{a_i}{a_j} \in \mathbb{R}\{0\} \). So the structure group is \( \mathbb{R}\{0\} \) with multiplication.

For an \( n \)-dimensional manifold \( M \), the structure group of \( TM \) is \( GL(n, \mathbb{R}) \).

• A fiber bundle where the standard fiber is a vector space is called a vector bundle.

A cylinder can be made by gluing two opposite edges of a flat strip of paper. This is then a Cartesian product of a circle \( S^1 \) with a line segment \( I \). So \( B = S^1, F = I \) and this is a trivial bundle, i.e. globally a product space. On the other hand, a Möbius strip is obtained by twisting the strip and then gluing. Locally for some open set \( U \subset S^1 \) we can still write a segment of the Möbius strip as \( U \times I \), but the total space is no longer a product space. As a bundle, the Möbius strip is non-trivial.

• Given two bundles \((E_1, \pi_1, B_1)\) and \((E_2, \pi_2, B_2)\), the relevant or useful maps between these are those which preserve the bundle structure locally, i.e. those which map fibers into fibers. They are called bundle morphisms. □

A bundle morphism is a pair of maps \((F,f) : F : E_1 \rightarrow E_2, f : \)
$B_1 \rightarrow B_2$, such that $\pi_2 \circ F = f \circ \pi_1$. (This is of course better understood in terms of a commutative diagram.)

Not all systems of coordinates are appropriate for a bundle. But it is possible to define a set of fiber coordinates in the following way. Given a differentiable fiber bundle with $n$-dimensional base manifold $B$ and $p$-dimensional fiber $F$, the coordinates of the bundle are given by bundle morphisms onto open sets of $\mathbb{R}^n \times \mathbb{R}^p$.

- Given a manifold $M$ the tangent space $T_PM$, consider $A_P = (e_1, \cdots, e_n)$, a set of $n$ linearly independent vectors at $P$. $A_P$ is a basis in $T_PM$. The typical fiber in the frame bundle is the set of all bases, $F = \{A_P\}$. 

Given a particular basis $\bar{A}_P = (\bar{e}_1, \cdots, \bar{e}_n)$, any basis $A_P$ may be expressed as

$$e_i = a^j_i \bar{e}_j. \quad (23.1)$$

The numbers $a^j_i$ can be thought of as the components of a matrix, which must be invertible so that we can recover the original basis from the new one. Thus, starting from any one basis, any other basis can be reached by an $n \times n$ invertible matrix, and any $n \times n$ invertible matrix produces a new basis. So there is a bijection between the set of all frames in $T_PM$ and $GL(n, \mathbb{R})$.

Clearly the structure group of the typical fiber of the frame bundle is also $GL(n, \mathbb{R})$.

- A fiber bundle in which the typical fiber $F$ is identical (or homeomorphic) to the structure group $G$, and $G$ acts on $F$ by left translation is called a principal fiber bundle.

  **Example:**
  1. Typical fiber = $S^1$, structure group $U(1)$.
  2. Typical fiber = $S^3$, structure group $SU(2)$.
  3. Bundle of frames, for which the typical fiber is $GL(n, \mathbb{R})$, as is the structure group.

- A section of a fiber bundle $(E, \pi, B)$ is a mapping $s : B \rightarrow E, p \mapsto s(p)$, where $p \in B, s(p) \in \pi^{-1}(p)$. So we can also say $\pi \circ s = \text{identity}$. 

  **Example:** A vector field is a section of the tangent bundle, $v : P \mapsto v_P$.

  **Example:** A function on $M$ is a section of the bundle which locally looks like $M \times \mathbb{R}$ (or $M \times \mathbb{C}$ if we are talking about complex functions).
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- Starting from the tangent bundle we can define the **cotangent bundle**, in which the typical fiber is the dual space of the tangent space. This is written as $T^*\mathcal{M}$. As we have seen before, a section of $T^*\mathcal{M}$ is a 1-form field on $\mathcal{M}$.

- Remember that a **vector bundle** $F \to E \xrightarrow{\pi} B$ is a bundle in which the typical fiber $F$ is a vector space.

- A vector bundle $(E, \tilde{\pi}, B, F, G)$ with typical fiber $F$ and structure group $G$ is said to be **associated** to the principal bundle $(P, \pi, B, G)$ by the representation $\{D(g)\}$ of $G$ on $F$ if its transition functions are the images under $D$ of the transition functions of $P$.

  That is, suppose we have a covering $\{U_i\}$ of $B$, and local trivialization of $P$ with respect to this covering is $\Phi_i: \pi^{-1}(U_i) \to U_i \times G$, which is essentially the same as writing $\Phi_{i,x}: \pi^{-1}(x) \to G$, $x \in U_i$. Then the transition functions of $P$ are of the form

  $$g_{ij} = \Phi_i \circ \Phi_j^{-1}: U_i \cap U_j \to G.$$  

  The transition functions of $E$ corresponding to the same covering of $B$ are given by $\phi_i: \tilde{\pi}^{-1}(U_i) \to U_i \times F$ with $\phi_i \circ \phi_j^{-1} = D(g_{ij})$.

  That is, if $v_i$ and $v_j$ are images of the same vector $v_x \in F_x$ under overlapping trivializations $\phi_i$ and $\phi_j$, we must have

  $$v_i = D(g_{ij}(x)) v_j.$$  

  A more physical way of saying this is that if two observers look at the same vector at the same point, their observations are related by a group transformation $(p, v) \simeq (p, D(g_{ij}v))$.

- These relations are what are called **gauge transformations** in physics, and $G$ is called the **gauge group**. Usually $G$ is a Lie group for reasons of continuity.

  Fields appearing in various physical theories are sections of vector bundles, which in some trivialization look like $U_\alpha \times V$ where $U_\alpha$ is some open neighborhood of the point we are interested in, and $V$ is a vector space. $V$ carries a representation of some group $G$, usually a Lie group, which characterizes the theory.

  To discuss this a little more concretely, let us consider an associated vector bundle $(E, \tilde{\pi}, B, F, G)$ of a principal bundle $(P, \pi, B, G)$. Then the transition functions are in some representation of the group $G$. Because the fiber carries a representation $\{D(g)\}$ of $G$, there are
always linear transformations $T_x : E_x \to E_x$ which are members of the representation $\{D(g)\}$. Let us write the space of all sections of this bundle as $\Gamma(E)$. An element of $\Gamma(E)$ is a map from the base space to the bundle. Such a map assigns an element of $V$ to each point of the base space.

We say that a linear map $T : \Gamma(E) \to \Gamma(E)$ is a **gauge transformation** if at each point $x$ of the base space, $T_x \in \{D(g)\}$ for some $g$, i.e. if

$$T_x : (x, v)_\alpha \mapsto (x, D(g)v)_\alpha ,$$

(23.4)

for some $g \in G$ and for $(x, v)_\alpha \in U_\alpha \times F$. In other words, a gauge transformation is a representation-valued linear transformation of the sections at each point of the base space. The right hand side is often written as $(x, gv)_\alpha$.

This definition is independent of the choice of $U_\alpha$. To see this, consider a point $x \in U_\alpha \cap U_\beta$. Then

$$(x, v)_\alpha = (x, g_{\beta\alpha}v)_\beta .$$

(23.5)

In the other notation we have been using, $v_\alpha$ and $v_\beta$ are images of the same vector $v_x \in V_x$, and $v_\beta = D(g_{\beta\alpha})v_\alpha$. A gauge transformation $T$ acts as

$$T_x : (x, v)_\alpha \mapsto (x, gv)_\alpha .$$

(23.6)

But we also have

$$(x, gv)_\alpha = (x, g_{\beta\alpha}gv)_\beta$$

(23.7)

using Eq. (23.5). So it is also true that

$$T_x : (x, g_{\beta\alpha}v)_\beta \mapsto (x, g_{\beta\alpha}gv)_\beta .$$

(23.8)

Since $F$ carries a representation of $G$, we can think of $gv$ as a change of variables, i.e. define $v' = g_{\beta\alpha}v$. Then Eq. (23.8) can be written also as

$$T_x : (x, v')_\beta \mapsto (x, g'v')_\beta ,$$

(23.9)

where now $g' = g_{\beta\alpha}gg_{\beta\alpha}^{-1}$. So $T$ is a gauge transformation in $U_\beta$ as well. The definition of a gauge transformation is independent of the
choice of $U_\alpha$, but $T$ itself is not. The set of all gauge transformations $\mathcal{G}$ is a group, with

$$(gh)(x) = g(x)h(x), \quad (g^{-1})(x) = (g(x))^{-1}. \quad (23.10)$$

- The groups $G$ and $\mathcal{G}$ are both called the \textbf{gauge group} by different people. \qed