Chapter 15

Volume form

The space of p-forms in n dimensions is $\binom{n}{p}$ dimensional. So the space of n-forms in n dimensions is 1-dimensiona, i.e., there is only one independent component, and all n-forms are scalar multiples of one another.

Choose an n-form field. Call it ω. Suppose $\omega \neq 0$ at some point P. Then given any basis $\{e_\mu\}$ of $T_P\mathcal{M}$, we have $\omega(e_1, \cdots e_n) \neq 0$ since $\omega \neq 0$. Thus all vector bases at P fall into two classes, one for which $\omega(e_1, \cdots e_n) > 0$ and the other for which it is < 0.

Once we have identified these two classes, they are independent of ω. That is, if ω' is another n-form which is non-zero at P, there must be some function $f \neq 0$ such that $\omega' = f\omega$. Two bases which gave positive numbers under ω will give the same sign — both positive or both negative — under ω' and therefore will be in the same class.

• So every basis (set of n linearly independent vectors) is a member of one of the two classes. These are called **righthanded** and **lefthanded**. □

• A manifold is called **orientable** if it is possible to define a continuous n-form field ω which is non-zero everywhere on the manifold. Then it is possible to choose a basis with the same handedness everywhere on the manifold continuously. □

Euclidean space is orientable, the Möbius band is not.

• An orientable manifold is called **oriented** once an **orientation** has been chosen, i.e. once we have decided to choose basis vectors with the same handedness everywhere on the manifold. □

• It is necessary to choose an oriented manifold when we discuss the integration of forms. On an n-dimensional manifold, a set of n

linearly independent vectors define an n-dimensional parallelepiped. If we define an n-form $\omega \neq 0$ we can think of the value of these vectors as the volume of this parallelepiped. This ω is called a **volume form**. □

Once a volume form has been chosen, any set of n linearly independent vectors will define a positive or negative volume.

The integral of a function f on $\mathbb{R}^n$ is the sum of the values of f, multiplied by infinitesimal volumes of coordinate elements. Similarly, we define the integral of a function f on an oriented manifold as the sum of the values of f, multiplied by infinitesimal volumes. The way to do that is the following.

Given a function f, define an n-form in a chart by $\omega = f dx^1 \wedge \cdots \wedge dx^n$. To integrate over an open set U, divide it up into infinitesimal 'cells', spanned by vectors

$$\left\{ \Delta x^1 \frac{\partial}{\partial x^1}, \Delta x^2 \frac{\partial}{\partial x^2}, \cdots, \Delta x^n \frac{\partial}{\partial x^n} \right\},$$

where the Δx^i are small numbers.

Then the integral of f over one such cell is approximately

$$\begin{aligned} f \Delta x^1 \Delta x^2 \cdots \Delta x^n &= f dx^1 \wedge \cdots \wedge dx^n \left(\Delta x^1 \partial_1, \cdots, \Delta x^n \partial_n \right) \\ &= \omega(cell)\,. \end{aligned} \tag{15.1}$$

Adding up the contributions from all cells and taking the limit of cell size going to zero, we find

$$\int\limits_U \omega = \int\limits_{\varphi(U)} f d^n x\,. \tag{15.2}$$

The right hand side is the usual integration in calculus of n variables, and the left hand side is our notation which we are defining.

The right hand side can be seen to be independent of the choice of coordinate system. If we choose a different coordinate system, we get a Jacobian, but also a redefinition of the region $\varphi(U)$. Let us check that the left hand side is also invariant of the choice of the coordinates. We will do this in two dimensions with $\omega = f dx^1 \wedge dx^2$.

In another coordinate system (y^1, y^2) corresponding to $\varphi'(U)$

$$\begin{aligned} dx^1 &= \frac{\partial x^1}{\partial y^1}\, dy^1 + \frac{\partial x^1}{\partial y^2}\, dy^2 \\ dx^2 &= \frac{\partial x^2}{\partial y^1}\, dy^1 + \frac{\partial x^2}{\partial y^2}\, dy^2 \\ \Rightarrow \qquad dx^1 \wedge dx^2 &= \left(\frac{\partial x^1}{\partial y^1}\frac{\partial x^2}{\partial y^1} - \frac{\partial x^1}{\partial y^2}\frac{\partial x^2}{\partial y^2} \right) dy^1 \wedge dy^2 \\ &= J dy^1 \wedge dy^2 \,, \end{aligned} \tag{15.3}$$

and J is the Jacobian.

So what we have here is

$$\begin{aligned} \int\limits_U \omega &= \int\limits_U f(x^1, x^2) dx^1 \wedge dx^2 \\ &= \int\limits_U f(y^1, y^2) J dy^1 \wedge dy^2 \\ &= \int\limits_{\varphi'(U)} f(y^1, y^2) J d^2 y \,, \end{aligned} \tag{15.4}$$

so we get the same result both ways.

Given the same f, if we choose a basis with the opposite orientation, the integral of ω will have the opposite sign. This is why the choice of orientation has to be made before integration.

Manifolds become even more interesting if we define a metric.