Chapter 14

Exterior derivative

The exterior derivative is a generalization of the gradient of a function. It is a map from \( p \)-forms to \((p+1)\)-forms. This should be a derivation, so it should be linear,

\[
d(\alpha + \omega) = d\alpha + d\omega \quad \forall \text{forms } \alpha, \omega.
\] (14.1)

This should also satisfy Leibniz rule, but the algebra of \( p \)-forms is not a commutative algebra but a graded commutator algebra, i.e., involves a factor of \((-1)^{pq}\) for exchanges,

\[
\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha,
\] (14.2)

as we have seen. We wish to define the exterior derivative so that it is compatible with this property, i.e.,

\[
d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{pq} d\beta \wedge \alpha.
\] (14.3)

Alternatively we can write

\[
d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.
\] (14.4)

This will be the Leibniz rule for wedge products. Note that it gives the correct result when one or both of \( \alpha, \beta \) are 0-forms, i.e., functions. The two formulas are identical by virtue of the fact that \( d\beta \) is a \((q+1)\)-form, so that

\[
\alpha \wedge d\beta = (-1)^{p(q+1)} d\beta \wedge \alpha.
\] (14.5)

We will try to define the exterior derivative in a way such that it has these properties.
Let us define the exterior derivative of a $p$-form $\omega$ in a chart as

$$d\omega = \frac{1}{p!} \partial_i \omega_{i_1 \cdots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$$  \hspace{1cm} (14.6)$$

This clearly has the first property of linearity. To check the (graded) Leibniz rule, let us write $\alpha \wedge \beta$ in components. Then

$$d(\alpha \wedge \beta) = \frac{1}{p!q!} \partial_i (\alpha_{i_1 \cdots i_p} \beta_{j_1 \cdots j_q}) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$$

$$= \frac{1}{p!} \left[ (\partial_i \alpha_{i_1 \cdots i_p}) \beta_{j_1 \cdots j_q} + \alpha_{i_1 \cdots i_p} (\partial_i \beta_{j_1 \cdots j_q}) \right] dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$$

$$= \frac{1}{p!q!} \left( \partial_i \alpha_{i_1 \cdots i_p} \beta_{j_1 \cdots j_q} + \alpha_{i_1 \cdots i_p} \partial_i \beta_{j_1 \cdots j_q} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q}$$

$$+ \frac{1}{p!q!} (-1)^p \alpha_{i_1 \cdots i_p} \partial_i \beta_{j_1 \cdots j_q} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q}$$

$$= d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.$$  \hspace{1cm} (14.7)

A third property of the exterior derivative immediately follows from here,

$$d^2 = 0.$$  \hspace{1cm} (14.8)

To see this, we write

$$d(d\omega) = \frac{1}{p!} d \left( \partial_i \omega_{i_1 \cdots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p} \right)$$

$$= \frac{1}{p!} \partial_j \partial_i \omega_{i_1 \cdots i_p} dx^j \wedge dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \hspace{1cm} (14.9)$$

But the wedge product is antisymmetric, $dx^j \wedge dx^i = -dx^i \wedge dx^j$, and the indices are summed over, so the above object must be antisymmetric in $\partial_j, \partial_i$. But that vanishes. So $d^2 = 0$ on all forms.

Note that we can also write

$$d\omega = \frac{1}{p!} \left( \partial_i \omega_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \right),$$  \hspace{1cm} (14.10)

where the object in parentheses is a gradient 1-form corresponding to the gradient of the component.

Consider a 1-form $A = A_\mu dx^\mu$ where $A_\mu$ are smooth functions on $M$. Then using this definition we can write

$$dA = (dA_\nu) \wedge dx^\nu$$

$$= \partial_\mu A_\nu dx^\mu \wedge dx^\nu$$

$$= \frac{1}{2} (\partial_\nu A_\mu - \partial_\mu A_\nu) dx^\mu \wedge dx^\nu$$

$$\Rightarrow (dA)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  \hspace{1cm} (14.11)
We can generalize this result to write for a $p$-form,

$$\alpha = \frac{1}{p!} \alpha_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \quad (14.12)$$

$$d\alpha = \frac{1}{p!} (d\alpha_{\mu_1 \ldots \mu_p}) \ dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

$$= \frac{1}{(p+1)!} \partial_{[\mu} \alpha_{\nu_1 \ldots \nu_{p+1}]} dx^{\mu} \wedge dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{p+1}}$$

$$\Rightarrow \ (d\alpha)_{\mu_1 \ldots \mu_p} = \partial_{[\mu} \alpha_{\nu_1 \ldots \nu_p]} \quad (14.13)$$

**Example:** For $p = 1$ i.e. for a 1-form $A$ we get from this formula $(dA)_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$, in agreement with our previous calculation.

For $p = 2$ we have a 2-form, call it $\alpha$. Then using this formula we get

$$(d\alpha)_{\mu \nu \lambda} = \partial_{[\mu} \alpha_{\nu \lambda]}$$

$$= \partial_{\mu} \alpha_{\nu \lambda} - \partial_{\nu} \alpha_{\mu \lambda} + \partial_{\nu} \alpha_{\lambda \mu} - \partial_{\lambda} \alpha_{\mu \nu} + \partial_{\lambda} \alpha_{\nu \mu} - \partial_{\mu} \alpha_{\lambda \nu}.$$  

(14.14)

Note that $d$ is not defined on arbitrary tensors, but only on forms.

By definition, $d^2 = 0$ on any $p$-form. So if $\alpha = d\beta$, it follows that $d\alpha = 0$. But given a $p$-form $\alpha$ for which $d\alpha = 0$, can we say that there must be some $(p-1)$-form $\beta$ such that $\alpha = d\beta$?

- This is a good place to introduce some terminology. Any form $\omega$ such that $d\omega = 0$ is called **closed**, whereas any form $\alpha$ such that $\alpha = d\beta$ is called **exact**.

So every exact form is closed. Is every closed form exact? The answer is yes, in a sufficiently small neighbourhood. We say that every closed form is locally exact. Note that if a $p$-form $\alpha = d\beta$, we cannot uniquely specify the $(p-1)$-form $\beta$ since for any $(p-2)$-form $\gamma$, we can always write $\alpha = d\beta'$, where $\beta' = \beta + d\gamma$.

Thus a more precise statement is that given any $p$-form $\alpha$ such that $d\alpha = 0$ in a neighbourhood of some point $P$, there is some neighbourhood of this point and some $(p-1)$-form $\beta$ such that $\alpha = d\beta$ in that neighbourhood. But this may not be true globally. This statement is known as the **Poincaré lemma**.

**Example:** In $\mathbb{R}^2$ remove the origin. Consider the 1-form

$$\alpha = \frac{xdy - ydx}{x^2 + y^2}.$$  

(14.15)
Then
\[
d\alpha = \left( \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} \right) dx \wedge dy - \left( \frac{1}{x^2 + y^2} - \frac{2y^2}{(x^2 + y^2)^2} \right) dy \wedge dx
\]
\[
= \frac{2}{x^2 + y^2} dx \wedge dy - \frac{2}{(x^2 + y^2)^2} x^2 + y^2 dx \wedge dy = 0.
\] (14.16)

Introduce polar coordinates \( r, \theta \) with \( x = r \cos \theta, y = r \sin \theta \).

Then
\[
dx = dr \cos \theta - r \sin \theta d\theta \quad dy = dr \sin \theta + r \cos \theta d\theta
\]
\[
\alpha = \frac{r \cos \theta (\sin \theta dr + r \cos \theta d\theta)}{r^2} - \frac{r \sin \theta (\cos \theta dr - r \sin \theta d\theta)}{r^2}
\]
\[
= \frac{r^2 (\cos^2 \theta + \sin^2 \theta) d\theta}{r^2} = d\theta.
\] (14.17)

Thus \( \alpha \) is exact, but \( \theta \) is multivalued so there is no function \( f \) such that \( \alpha = df \) everywhere. In other words, \( \alpha = d\theta \) is exact only in a neighbourhood small enough that \( \theta \) remains single-valued.