Chapter 13

Differential forms

There is a special class of tensor fields, which is so useful as to have a separate treatment. There are called **differential $p$-forms** or **$p$-forms** for short.

- A **$p$-form** is a $(0,p)$ tensor which is completely antisymmetric, i.e., given vector fields $v_1, \ldots, v_p$,

$$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_p) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_p)$$

(13.1)

for any pair $i, j$.

A 0-form is defined to be a function, i.e. an element of $C^\infty(M)$, and a 1-form is as defined earlier.

The antisymmetry of any $p$-form implies that it will give a non-zero result only when the $p$ vectors are linearly independent. On the other hand, no more than $n$ vectors can be linearly independent in an $n$-dimensional manifold. So $p \leq n$.

Consider a 2-form $A$. Given any two vector fields $v_1, v_2$, we have $A(v_1, v_2) = -A(v_2, v_1)$. Then the components of $A$ in a chart are

$$A_{ij} = A(\partial_i, \partial_j) = -A_{ji}.$$ 

(13.2)

Similarly, for a $p$-form $\omega$, the components are $\omega_{i_1 \ldots i_p}$, and components are multiplied by $(-1)$ whenever any two indices are interchanged.

It follows that a $p$-form has $\binom{n}{p}$ independent components in $n$-dimensions.

Any 1-form produces a function when acting on a vector field. So given a pair of 1-forms $A, B$, it is possible to construct a 2-form $\omega$.
by defining
\[ \omega(u, v) = A(u)B(v) - B(u)A(v), \quad \forall u, v. \quad (13.3) \]

- This is usually written as \( \omega = A \otimes B - B \otimes A \), where \( \otimes \) is called the outer product.

- Then the above construction defines a product written as
\[ \omega = A \wedge B = -B \wedge A, \quad (13.4) \]
and called the wedge product. Clearly, \( \omega \) is a 2-form.

Let us work in a coordinate basis, but the results we find can be generalized to any basis. The coordinate bases for the vector fields, \( \{ \partial_i \} \), and 1-forms, \( \{ dx^i \} \), satisfy \( dx^i(\partial_j) = \delta^i_j \). A 1-form \( A \) can be written as \( A = A_i dx^i \), and a vector field \( v \) can be written as \( v = v^i \partial_i \), so that \( A(v) = A_i v^i \). Then for the \( \omega \) defined above and for any pair of vector fields \( u, v \),
\[ \omega(u, v) = A(u)B(v) - B(u)A(v) = A_i u^i B_j v^j - B_i u^i A_j v^j = (A_i B_j - B_i A_j) u^i v^j. \quad (13.5) \]

The components of \( \omega \) are \( \omega_{ij} = \omega(\partial_i, \partial_j) \), so that
\[ \omega(u, v) = \omega(u^i \partial_i, v^j \partial_j) = \omega_{ij} u^i v^j. \quad (13.6) \]

Then \( \omega_{ij} = A_i B_j - B_i A_j \) for the 2-form defined above. We can now construct a basis for 2-forms, which we write as \( dx^i \wedge dx^j \),
\[ dx^i \wedge dx^j = dx^i \otimes dx^j - dx^j \otimes dx^i. \quad (13.7) \]

Then a 2-form can be expanded in this basis as
\[ \omega = \frac{1}{2!} \omega_{ij} dx^i \wedge dx^j, \quad (13.8) \]
because then
\[ \omega(u, v) = \frac{1}{2!} \omega_{ij} (dx^i \otimes dx^j - dx^j \otimes dx^i) (u, v) \]
\[ = \frac{1}{2!} \omega_{ij} (u^i v^j - u^j v^i) = \omega_{ij} u^i v^j. \quad (13.9) \]
Similarly, a basis for $p$–forms is

$$dx^{i_1} \wedge \cdots \wedge dx^{i_p} = dx^{[i_1} \otimes \cdots \otimes dx^{i_p]}, \quad (13.10)$$

where the square brackets stand for total antisymmetrization: all even permutations of the indices are added and all the odd permutations are subtracted. (Caution: some books define the ‘square brackets’ as antisymmetrization with a factor $1/p!$. ) For example, for a 3-form, a basis is

$$dx^i \wedge dx^j \wedge dx^k = dx^i \otimes dx^j \otimes dx^k - dx^j \otimes dx^i \otimes dx^k$$

$$+ dx^j \otimes dx^k \otimes dx^i - dx^k \otimes dx^i \otimes dx^j$$

$$+ dx^k \otimes dx^i \otimes dx^j - dx^i \otimes dx^k \otimes dx^j. \quad (13.11)$$

Then an arbitrary 3-form $\Omega$ can be written as

$$\Omega = \frac{1}{3!} \Omega_{ijk} dx^i \wedge dx^j \wedge dx^k. \quad (13.12)$$

Note that there is a sum over indices, so that the factorial goes away if we write each basis 3-form up to permutations, i.e. treating different permutations as equivalent. Thus a $p$–form $\alpha$ can be written in terms of its components as

$$\alpha = \frac{1}{p!} \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \quad (13.13)$$

**Examples:** A 2-form in two dimensions can be written as

$$\omega = \frac{1}{2!} \omega_{ij} dx^i \wedge dx^j$$

$$= \frac{1}{2!} (\omega_{12} dx^1 \wedge dx^2 + \omega_{21} dx^2 \wedge dx^1)$$

$$= \frac{1}{2!} (\omega_{12} - \omega_{21}) dx^1 \wedge dx^2$$

$$= \omega_{12} dx^1 \wedge dx^2. \quad (13.14)$$

A 2-form in three dimensions can be written as

$$\omega = \frac{1}{2!} \omega_{ij} dx^i \wedge dx^j$$

$$= \omega_{12} dx^1 \wedge dx^2 + \omega_{23} dx^2 \wedge dx^3 + \omega_{31} dx^3 \wedge dx^1 \quad (13.15)$$
In three dimensions, consider two 1-forms \( \alpha = \alpha_i dx^i \), \( \beta = \beta_i dx^i \). Then
\[
\alpha \wedge \beta = (\alpha_i \beta_j - \alpha_j \beta_i) \frac{1}{2!} dx^i \wedge dx^j \\
= \alpha_i \beta_j dx^i \wedge dx^j \\
= (\alpha_1 \beta_2 - \alpha_2 \beta_1) dx^1 \wedge dx^2 \\
+ (\alpha_2 \beta_3 - \alpha_3 \beta_2) dx^2 \wedge dx^3 \\
+ (\alpha_3 \beta_1 - \alpha_1 \beta_3) dx^3 \wedge dx^1.
\]
(13.16)

The components are like the cross product of vectors in three dimensions. So we can think of the wedge product as a generalization of the cross product.

- We can also define the **wedge product** of a \( p \)-form \( \alpha \) and a \( q \)-form \( \beta \) as a \( (p + q) \)-form satisfying, for any \( p + q \) vector fields \( v_1, \cdots, v_{p+q} \),
\[
\alpha \wedge (v_1, \cdots, v_{p+q}) = \frac{1}{p!q!} \sum_P (-1)^{\text{deg } P} \alpha \otimes \beta (P(v_1, \cdots, v_{p+q})) .
\]
(13.17)

Here \( P \) stands for a permutation of the vector fields, and \( \text{deg } P \) is 0 or 1 for even and odd permutations, respectively. In the outer product on the right hand side, \( \alpha \) acts on the first \( p \) vector fields in a given permutation \( P \), and \( \beta \) acts on the remaining \( q \) vector fields.

The wedge product above can also be defined in terms of the components of \( \alpha \) and \( \beta \) in a chart as follows.
\[
\alpha = \frac{1}{p!} \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\
\beta = \frac{1}{q!} \beta_{j_1 \cdots j_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q} \\
\alpha \wedge \beta = \frac{1}{p!q!} \alpha_{i_1 \cdots i_p} \beta_{j_1 \cdots j_q} (dx^{i_1} \wedge \cdots \wedge dx^{i_p}) \wedge (dx^{j_1} \wedge \cdots \wedge dx^{j_q}) .
\]
(13.18)

Note that \( \alpha \wedge \beta = 0 \) if \( p + q > n \), and that a term in which some \( i \) is equal to some \( j \) must vanish because of the antisymmetry of the wedge product.

It can be shown by explicit calculation that wedge products are associative,
\[
\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma .
\]
(13.19)
Cross-products are not associative, so there is a distinction between cross-products and wedge products. In fact, for 1-forms in three dimensions, the above equation is analogous to the identity for the triple product of vectors,

\[ a \cdot (b \times c) = (a \times b) \cdot c. \]  

(13.20)

For a \( p \)-form \( \alpha \) and \( q \)-form \( \beta \), we find

\[ \alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha. \]  

(13.21)

**Proof:** Consider the wedge product written in terms of the components. We can ignore the parentheses separating the basis forms since the wedge product is associative. Then we exchange the basis 1-forms. One exchange gives a factor of \(-1\),

\[ dx^{i_1} \wedge dx^{i_2} = -dx^{i_2} \wedge dx^{i_1}. \]  

(13.22)

Continuing this process, we get

\[ dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q} \]
\[ = (-1)^p dx^{j_1} \wedge dx^{i_1} \wedge \cdots \wedge dx^{j_p} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_q} \]
\[ = \cdots \]
\[ = (-1)^{pq} dx^{j_1} \wedge \cdots \wedge dx^{j_q} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \]  

(13.23)

Putting back the components, we find

\[ \alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \]  

(13.24)
as wanted.

- The wedge product defines an algebra on the space of differential forms. It is called a **graded commutative algebra**.
- Given a vector field \( v \), we can define its **contraction** with a \( p \)-form by

\[ \iota_v \omega = \omega(v, \cdots) \]  

(13.25)

with \( p - 1 \) empty slots. This is a \((p - 1)\)-form. Note that the position of \( v \) only affects the sign of the contracted form.

**Example:** Consider a 2-form made of the wedge product of two 1-forms, \( \omega = \lambda \wedge \mu = \lambda \otimes \mu - \mu \otimes \lambda \). Then contraction by \( v \) gives

\[ \iota_v \omega = \omega(v, \bullet) = \lambda(v)\mu - \mu(v)\lambda = -\omega(\bullet, v). \]  

(13.26)
If we have a $p$-form $\omega = \frac{1}{p!} \omega_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$, its contraction with a vector field $v = v^i \partial_i$ is

$$\iota_v \omega = \frac{1}{(p-1)!} \omega_{i_1 \cdots i_{p-1} v^i} dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}} \wedge dx^{i_p}.$$  \hfill (13.27)

Note the sum over indices. To see how the factor becomes $\frac{1}{(p-1)!}$, we write the contraction as

$$\iota_v \omega = \frac{1}{p!} \omega_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} (v^i \partial_i).$$  \hfill (13.28)

Since the contraction is done in the first slot, so we consider the action of each basis 1-form $dx^i$ on $\partial_i$ by carrying $dx^i$ to the first position and then writing a $\delta_{i_k}^i$ for each exchange, but we get the same factor by rearranging the indices of $\omega$, thus getting a +1 for each index. This leads to an overall factor of $p$.

- given a diffeomorphism $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$, the \textbf{pullback} of a 1-form $\lambda$ (on $\mathcal{M}_2$) is $\varphi^* \lambda$, defined by

$$\varphi^* \lambda(v) = \lambda(\varphi_* v)$$  \hfill (13.29)

for any vector field $v$ on $\mathcal{M}_1$.

Then we can consider the pullback $\varphi^* dx^i$ of a basis 1-form $dx^i$. For a general 1-form $\lambda = \lambda_idx^i$, we have $\varphi^* \lambda = \varphi^*(\lambda_idx^i)$. But

$$\varphi^* \lambda(v) = \lambda(\varphi_* v) = \lambda_i dx^i(\varphi_* v).$$  \hfill (13.30)

Now, $dx^i(\varphi_* v) = \varphi^* dx^i(v)$ and the thing on the right hand side is a function on $\mathcal{M}_1$, so we can write this as

$$\varphi^* \lambda(v) = (\varphi^* \lambda_i) \varphi^* dx^i(v),$$  \hfill (13.31)

where $\varphi^* \lambda_i$ are now functions on $\mathcal{M}_1$, i.e.

$$(\varphi^* \lambda_i)|_p = \lambda_i|_{\varphi(p)}.$$  \hfill (13.32)

So we can write $\varphi^* \lambda = (\varphi^* \lambda_i) \varphi^* dx^i$. For the wedge product of two 1-forms,

$$\varphi^* (\lambda \wedge \mu)(u, v) = (\lambda \wedge \mu)(\varphi_* u, \varphi_* v)$$

$$= \lambda \otimes \mu(\varphi_* u, \varphi_* v) - \mu \otimes \lambda(\varphi_* u, \varphi_* v)$$

$$= \lambda(\varphi_* u)\mu(\varphi_* v) - \mu(\varphi_* u)\lambda(\varphi_* v)$$

$$= \varphi^* \lambda(u)\varphi^* \mu(v) - \varphi^* \mu(u)\varphi^* \lambda(v)$$

$$= (\varphi^* \lambda \wedge \varphi^* \mu)(u, v).$$  \hfill (13.33)
Since $u, v$ are arbitrary vector fields it follows that

$$
\varphi^*(\lambda \wedge \mu) = \varphi^*\lambda \wedge \varphi^*\mu \\
\varphi^*(dx^i \wedge dx^j) = \varphi^*dx^i \wedge \varphi dx^j.
$$  \tag{13.34}

Since the wedge product is associative, we can write (by assuming an obvious generalization of the above formula)

$$
\varphi^* (dx^i \wedge dx^j \wedge dx^k) = \varphi^* \left( (dx^i \wedge dx^j) \wedge dx^k \right)
$$

$$
= \varphi^* (dx^i \wedge dx^j) \wedge \varphi^* dx^k
$$

$$
= \varphi^* dx^i \wedge \varphi^* dx^j \wedge \varphi^* dx^k,
$$  \tag{13.35}

and we can continue this for any number of basis 1-forms. So for any $p$-form $\omega$, let us define the pullback $\varphi^* \omega$ by

$$
\varphi^* \omega (v_1, \cdots, v_p) = \omega (\varphi^* v_1, \cdots, \varphi^* v_p),
$$  \tag{13.36}

and in terms of components, by

$$
\varphi^* \omega = \frac{1}{p!} \left( \varphi^* \omega_{i_1 \cdots i_p} \right) \varphi^* dx^{i_1} \wedge \cdots \wedge dx^{i_p}.
$$  \tag{13.37}

We assumed above that the pullback of the wedge product of a 2-form and a 1-form is the wedge product of the pullbacks of the respective forms, but it is not necessary to make that assumption – it can be shown explicitly by taking three vector fields and following the arguments used earlier for the wedge product of two 1-forms.

Then for any $p$-form $\alpha$ and $q$-form $\beta$ we can calculate from this that

$$
\varphi^* (\alpha \wedge \beta) = \varphi^* \alpha \wedge \varphi^* \beta.
$$  \tag{13.38}

Thus pullbacks commute with (are distributive over) wedge products.